The Paradox of Vote Trading: Effects of Decision Rules and Voting Strategies on Externalities*

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Introduction

Vote trading, or logrolling, is a concept familiar to students of legislative bodies. Such students of the formal properties of voting bodies—James M. Buchanan, Gordon Tullock and James S. Coleman, in particular—have extolled the strategy of exchanging votes in a legislative arena as a means by which individual members are able to maximize their total utility.[1] Their scenario is straightforward: a legislator who does not feel very strongly about an issue he would oppose should be willing to trade votes with another member in order to secure the passage of a motion he strongly favors, but which would otherwise be defeated. Exchanging of votes thus seems analogous to exchanging of commodities for money in a market economy. William H. Riker and Steven J. Brams, however, have demonstrated that, for a legislature of three or more members and at least six motions, some trades which appear to be individually rational do not maximize the total utility of the traders.[2] Instead, there is a "paradox of vote trading" in which the incentive for every pair of legislators to trade is present, but the effects of externalities (costs imposed on the nontrader for each pair of motions) reduces the total utility to each player from what he could expect if no trades were made.

Riker and Brams argue that the advocates of vote exchanges have failed to consider the effects of externalities.[3] An externality is simply any cost which is imposed on an individual as a result of actions not under his control. In the case of the paradox of vote trading, such externalities considered by Riker and Brams are always "external costs"—i.e., the trading behavior of two members of a three-person legislature always adversely affects the nontrader.[4] This is a feature of the system of majority voting, as we shall demonstrate below. Indeed, externalities may involve potential benefits to the nonparticipant. We shall present an extension of the paradox of vote trading to unanimity decision rules below in which externalities are indeed positive. Actually, the extension of the Riker-Brams paradox to unanimity rules is part of a broader theoretical question which we shall consider: under what conditions is a more stringent decision rule or a change in voting strategies Pareto optimal? Accordingly, after a consideration of the Riker-Brams argument, we shall extend their analysis and attempt a resolu-

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[1] See James M. Buchanan and Gordon Tullock, *The Calculus of Consent: Logical Foundations of Constitutional Democracy* (Ann Arbor: University of Michigan Press, 1962), chaps. 7–12; Tullock, "A Simple Algebraic Logrolling Model," *American Economic Review*, 60 (June, 1970), 419–426; James S. Coleman, "The Possibility of a Social Welfare Function," *American Economic Review*, 56 (December, 1966), 1105–1122; Coleman, "The Possibility of a Social Welfare Function: Reply," *American Economic Review*, 57 (December, 1967), pp. 1311–1317; Edwin T. Haefele, "Coalitions, Minority Representation, and Vote-Trading Probabilities," *Public Choice*, 8 (Spring, 1970), 75–90; and Dennis C. Mueller, Geoffrey C. Philpotts, and Jaroslav Vanek, "The Social Gains from Exchanging Votes: A Simulation Approach," *Public Choice*, 13 (Fall, 1972), 55–80.

[2] Riker and Brams, "The Paradox of Vote Trading," *American Political Science Review*, 67 (December, 1973), 1235–1247. Cf. R. E. Park, "The Possibility of a Social Welfare Function: Comment," *American Economic Review*, 57 (December, 1967), 1300–1304; Mueller, "The Possibility of a Social Welfare Function; Comment," *Ibid.*, pp. 1304–1311; Robert Wilson, "An Axiomatic Model of Logrolling," *American Economic Review*, 59 (June, 1969), pp. 331–341; David H. Koehler, "Vote-Trading and the Voting Paradox: A Proof of Equivalence," *American Political Science Review*, 69 (1975); and, for a view which tends to support Riker and Brams but does not rule out the positions of Buchanan, Tullock and Coleman, see Peter Bernholz, "Logrolling, Arrow Paradox and Cyclical Majorities," *Public Choice*, 15 (Summer, 1973), 87–95 (but especially p. 94). But cf. the exchange among Tullock, Bernholz, and Riker and Brams in "Communications," *American Political Science Review*, 68 (December, 1974), 1688–1692. Also see Joe A. Oppenheimer, "Relating Coalitions of Minorities to the Voters' Paradox, or Putting the Fly in the Democratic Pie," University of Texas mimeo, 1973. An earlier informal statement of the possible negative effect of vote trading is found in William A. Niskanen, Jr., *Bureaucracy and Representative Government* (Chicago: Aldine, 1971), p. 143.

[3] Riker and Brams, "The Paradox of Vote Trading," p. 1240.

[4] *Ibid.*, p. 1242.

Table 1. Utilities of Members for Outcomes of Voting

	Preferred Outcome	Motion x $u_i(X)$	$u_i(\tilde{X})$	Preferred Outcome	Motion y $u_i(Y)$	$u_i(\tilde{Y})$
	Col 1	Col 2	Col 3	Col 4	Col 5	Col 6
*Member 1	X	1	−2	Y	1	−2
Member 2	X	1	−1	$\tilde{Y}$	−2	2
Member 3	$\tilde{X}$	−2	2	Y	1	−1
		Motion w $u_i(W)$	$u_i(\tilde{W})$		Motion z $u_i(Z)$	$u_i(\tilde{Z})$
Member 1	W	1	−1	$\tilde{Z}$	−2	2
Member 2	$\tilde{W}$	−2	2	Z	1	−1
*Member 3	W	1	−2	Z	1	−2
		Motion t $u_i(T)$	$u_i(\tilde{T})$		Motion v $u_i(V)$	$u_i(\tilde{V})$
Member 1	$\tilde{T}$	−2	2	V	1	−1
*Member 2	T	1	−2	V	1	−2
Member 3	T	1	−1	$\tilde{V}$	−2	2

* Nontrader on pair of motions.

tion on the supposed effects of vote trading between the Buchanan-Tullock-Coleman approach and that of Riker and Brams.

The paradox of vote trading under a majority decision rule is based upon a simple set of criteria for "naive" or "sincere" voting (always revealing one's actual preferences) and "sophisticated" voting (concealing actual preferences in a way which nevertheless will yield a net gain in expected utility).[5] The Riker-Brams model stipulates two conditions for "switching" (or, sophisticated voting): (1) a switcher must be in the decisive set for a given motion; and (2) he must also be a pivotal member of such a set—in the sense that his change in votes would be sufficient to make a formerly losing coalition a winning one.[6] The second condition contains the further premise that there is some positive utility in trading of votes, so that an incentive to employ sophisticated rather than sincere voting does exist. Vote trading is simply "a barter system of switching" in which each legislator who argees to trade his vote on one motion receives the vote of another member in return.[7] What Riker and Brams show, then, is that in an n-member legislature voting on a set of motions, trades which appear to be individually rational among each pair of voters are not collectively rational (i.e., Pareto optimal) because of the effects of externalities on the nontraders. An example drawn from the Riker-Brams article will demonstrate the paradox.

In Table 1, the utilities for each legislator on each motion which Riker and Brams stipulate are reproduced.[8] In the table we have six motions: x, y, w, z, t, and v and twelve possible outcomes: X, $\tilde{X}$, Y, $\tilde{Y}$, W, $\tilde{W}$, Z, $\tilde{Z}$, T, $\tilde{T}$, V, and $\tilde{V}$, where $u_i(X)$ is the utility to voter i of the passage of motion x and $u_i(\tilde{X})$ is the utility to i for the defeat of the motion. Utility is defined over outcomes, since the motions are merely means to the end of maximizing total utility for each individual. It is clear from the table that under simple majority rule without vote trading, the net utilities over the six motions are zero for each player.[9] While all six motions pass, variations in the intensity of preferences affect the net utility such that no one is any better or worse off than had no votes been cast at all. If each legislator ignores the actions of others on votes on which he himself is not in a position to trade, logrolling appears to be a rational strategy. The net expected utility of the six votes is +4 for each member, allowing each to make the best possible trades for himself.[10] If, on the other hand, Member 1 resigns himself to his "security level"—his evaluation of what happens to him if he refrains from trading but others do not (what he can guarantee himself regardless of the actions of other players)—his net utility would be −6.[11] In the former case, the member considers only the possible gains from vote trading, and in the latter case, only what happens to

[5] On the distinction between "sincere" and "sophisticated" voting, see Robin Farquharson, *Theory of Voting* (New Haven: Yale University Press, 1969), pp. 18 and 39–40.

[6] Riker and Brams, "The Paradox of Vote Trading," pp. 1237–1238.

[7] *Ibid.*, p. 1238.

[8] This is Table 5, p. 1241.

[9] *Ibid.*, Table 6.

[10] *Ibid.*, Table 7.

[11] *Ibid.*, Table 9.

Table 2. Utilities of Outcomes Without Trades Under a Unanimity Rule

	Outcomes						
Members	$u_i(X)$	$u_i(Y)$	$u_i(W)$	$u_i(Z)$	$u_i(T)$	$u_i(V)$	Σu_i
1	−2	−2	−1	2	2	−1	−2
2	−1	2	2	−1	−2	−2	−2
3	2	−1	−2	−2	−1	2	−2

him if he refuses to trade while the other players do trade. An evaluation of the potential gains from trading which excludes external costs convinces each member that all individual trades are rational. Thus, there is an incentive for every possible trade to occur. As a result of the logrolling, all six motions in Table 1 fail, yielding a net utility to each player of −2. This is the "paradox of vote trading." Each member fails to consider the external costs on the motions on which he does not trade and accepts any trade which appears to be individually rational. Rational trading thus yields a worse net situation than the failure to trade. Since no single member can be guaranteed that the others would agree not to trade if he refrains from doing so, each is put in a "kind of *n*-person Prisoners' Dilemma in which each member is forced by individual rationality to choose the socially and individually destructive alternative" and ". . . each member comes to a worse end than if he . . . always voted sincerely or naively."[12]

Riker and Brams have indeed revealed a paradox of rationality—one which they claim is "inescapable."[13] Advocates of the strategy of logrolling to maximize total utility—including Buchanan, Tullock, and Coleman—thus seem to be faced with a direct challenge to their position. In the next section of this paper, we shall extend the work of Riker and Brams to unanimity voting rules, providing further support for their finding of a paradox of vote trading.

Unanimity Decision Rules and the Paradox

The occurrence of the paradox of vote trading under simple majority rule is not sufficient to warrant the statement of Riker and Brams that the paradox is inescapable. As Buchanan and Tullock have argued, simple majority rule is not well suited to handle the problems posed by varying intensities of preference. An examination of Table 1 clearly reveals that the members of the legislative body are not equally affected by the outcomes on each motion. We assume that utilities are measured cardinally for each individual.[14] In this sense, our approach differs from that of Riker and Brams, who argue that only ordinal measurement of utilities is required to establish the paradox.[15] We shall justify this approach in the next section. For the present purposes, we need to note only that for each member, the net difference in utility which would result from the passage or failure of any given motion is maximized for those motions on which the member is in the minority position. Under such conditions of varying intensities of preference, Buchanan and Tullock argue the case for a system of qualified majority rule and go so far as to suggest that the optimal decision rule might be one of unanimity, allowing for vote trading.[16] On motion x for example, Members 1 and 2 each stand to gain one unit of utility if the motion passes, while Member 3 loses two units of utility with the same result. We might suspect that a more stringent voting rule would increase the net utility accruing to the three players more than would simple majority rule, either with or without vote trading.

The arguments of Buchanan and Tullock are, however, just as susceptible to the paradox of vote trading as is simple majority rule, although for quite different reasons. Under a unanimity rule without logrolling, the net utility over the six motions in Table 1 to each member is −2, with all motions failing. This result (see Table 2 below) is no better than majority rule with vote trading and inferior to majority rule without trading. If vote trading is allowed under unanimity, the net utility to each player is once again zero. (See Table 3). As in the case of simple majority rule, all six motions pass. Therefore, a unanimity rule allowing for vote trading is no better than simple majority rule with a prohibition on logrolling.[17] Furthermore, an examination of the effects of the alternative strategies of naive and sophisticated voting on the unanimity rule and utility values in Table 1 indicates that we have another occurrence of the paradox of vote trading. In the Riker-Brams example for majority rule, the individual trades were rational, but the aggregation of the

[12] *Ibid.*, pp. 1244 and 1242.
[13] *Ibid.*, p. 1242.
[14] This amounts to accepting interpersonal comparisons of utilities. See Jerome Rothenberg, *The Measurement of Social Welfare* (Englewood Cliffs, N.J.: Prentice-Hall, 1961), pp. 137–143.
[15] Riker and Brams, "The Paradox of Vote Trading," pp. 1242 ff.
[16] Buchanan and Tullock, *The Calculus of Consent*, pp. 85ff. and pp. 276–280.
[17] For a contrary point of view, see J. Roland Pennock, "The Pork Barrel and Majority Rule," Journal of Politics, 32 (August, 1970), 709–716.

Table 3. Utilities of Outcomes with Trades and with External Costs Under a Unanimity Rule

Members	Outcomes $u_i(X)$	$u_i(Y)$	$u_i(W)$	$u_i(Z)$	$u_i(T)$	$u_i(V)$	Σu_i
1	1	1	1	−2	−2	1	0
2	1	−2	−2	1	1	1	0
3	−2	1	1	1	1	−2	0

Table 4. Utilities of Outcomes with Trades When Possible and Without Anticipating External Costs Under a Unanimity Rule

Members	Outcomes $u_i(X)$	$u_i(Y)$	$u_i(W)$	$u_i(Z)$	$u_i(T)$	$u_i(V)$	Σu_i
1	−2	−2	1	−2	−2	1	−6
2	1	−2	−2	1	−2	−2	−6
3	−2	1	−2	−2	1	−2	−6

trading outcomes produced a sub-optimal societal result. For a unanimity rule, the trading of votes would produce a Pareto optimal societal (and, of course, individual) result, but, as we shall see, the individual trades are not themselves rational.

In the Riker-Brams example, the utility of the outcome to each legislator who trades but does not consider external costs is +4. For a unanimity decision rule, however, the failure to consider the impact of the trading behavior of others yields a utility of −6 to each member across the six motions (see Table 4). Thus, if each legislator considered only those trades in which he takes an active part, he would rationally refuse to trade. Consider the case of a potential exchange between members 2 and 3 on motions x and y. The only feasible trade would be for member 2 to give up his decisive opposition to motion y in return for member 3's pivotal support on x. In each case, however, the result would be a loss in utility. Voting sincerely across the two motions, each member would receive +1 units of utility: −1 on the defeated issue he supports and +2 on the defeated issue he opposes. An exchange of votes would yield each a net utility of −1: −2 on the passed motion he opposes and only +1 on the passed motion which he supports. It would appear, then, that there are no reasons why members should trade votes at all.

Such an appearance, however, is misleading because of the altered effects of externalities. Under a more exclusive voting rule, as Buchanan and Tullock correctly argue, the costs imposed bv externalities indeed will be reduced.[18] In the case of the current voting situation, logrolling will produce external benefits rather than costs for the traders. If each legislator were to refuse to trade and attempt to "await" the benefits accruing to him from the trading behavior of the other two legislators, he would receive (if such a situation were actually to occur) +8 units of utility—in comparison with the Riker-Brams security level of −6 for majority rule (see Table 5). While such trades will not take place—since they are individually irrational—the analogy with the Riker-Brams security level is instructive since it demonstrates that logrolling under a unanimity rule has a very different effect upon externalities than does vote trading under majority rule. This ideal situa-

[18] Buchanan and Tullock, *The Calculus of Consent*, pp. 63–68.

Table 5. Utilities of Outcomes When No Trades Occur But Each Member Anticipates External Costs Under a Unanimity Rule

Members	Outcomes $u_i(X)$	$u_i(Y)$	$u_i(W)$	$u_i(Z)$	$u_i(T)$	$u_i(V)$	Σu_i
1	1	1	1	2	2	1	8
2	1	2	2	1	1	1	8
3	2	1	1	1	1	2	8

Table 6. The Paradox of Vote Trading as a Three-Person Prisoners' Dilemma

Member 3	Alternative 1 Not Trade	Member 2: Alternative 1 Not Trade	Member 2: Alternative 2 Trade
Member 1	Alternative 1 Not Trade	−2, −2, −2	−2, −2, −2
Member 1	Alternative 2 Trade	−2, −2, −2	−4, −4, 8

Member 3	Alternative 2 Trade	Member 2: Alternative 1 Not Trade	Member 2: Alternative 2 Trade
Member 1	Alternative 1 Not Trade	−2, −2, −2	8, −4, −4
Member 1	Alternative 2 Trade	−4, 8, −4	0, 0, 0

Note: This is an adaptation of Table 10 in Riker and Brams, "The Paradox of Vote Trading," p. 1243. In cell (1, 1, 1), the entry is "−2, −2, −2" because no one trades. In cells (1, 2, 1), (2, 1, 1) and (1, 1, 2), the entries are also "−2, −2, −2" because, if any pair consistently votes sincerely, then all three must do so. In cell (2, 2, 2), the entry is "0, 0, 0" because all players trade. In cells (2, 2, 1), (1, 2, 2) and (2, 1, 2), the entries are appropriate combinations of −4 units for the traders and +8 units for the nontraders. The +8 units are calculated as in Table 5; the −4 units result from receiving, first the payoff to sincere voting (a) when one is not a trader (−2−2) and (b) when one's potential trade is with a sincere voter (1−2); and second, the payoff to sophisticated voting when one's partner is a sophisticated voter (2−1). If trading is viewed as "cooperative" behavior and not trading as "defecting," then the desired outcome in a Prisoners' Dilemma game—total cooperation—is represented in cell (2, 2, 2); this strategy is preferred by all players to complete defection—as represented in cell (1, 1, 1)—thus producing the dilemma.

tion which each player would like to obtain is thus simply the sum of all of the positive utilities for the six motions for each legislator: each member can guarantee himself a utility of +2 on those motions he opposes by refusing to trade; the trading patterns on the motions he favors guarantee that an outcome favorable to him will occur.

We thus have the reverse of the kind of *n*-person Prisoners' Dilemma found by Riker and Brams: each player would be better off if he could refuse to trade himself but would benefit from the external effects of the trades of the other players. Each member has an incentive to be a free rider on the others. By his single negative vote on a given motion he opposes, he can guarantee himself a utility of +2, while no trades will be consummated which will reject the motions he favors. Since, by construction, the nontrading member in Table 1 favors the motion on which he does not trade, he cannot incur any losses on such votes. Yet, if Members 1 and 3 were to realize that Member 2 did not intend to cooperate on other motions, they would rationally refuse to trade at all. The resulting paradox is that no players would ever trade, leaving each member worse off than he would be if the (individually irrational) trades were consummated. Table 6 presents the Prisoners' Dilemma for a unanimity rule.[19] The paradox does indeed seem "inescapable" and the entire situation is symmetric. The net utility to each legislator under simple majority rule without vote trading is equal to the net utility under a unanimity rule with vote trading; while the net utility obtained under both majority rule with vote trading and unanimity without logrolling is identical. These findings make more explicit a statement by E. E. Schattschneider about the bargaining system in American politics: ". . . if everybody got into

[19] Riker and Brams (see the note to Table 10 on p. 1243 of "The Paradox of Vote Trading") argue that their result is "strictly a Prisoners' Dilemma. . . ." If, however, we consider the general discussion of the dilemma, the preferred strategy of both players is cooperating (i.e., trading), but such a strategy is dominated by that of defecting (not trading). This is the case in our example; for a discussion of the dilemma, see R. Duncan Luce and Howard Raiffa, *Games and Decisions* (New York: John Wiley, 1957), pp. 88–113. The Riker-Brams example is indeed a Prisoners' Dilemma, but not the classical type.

the act, the unique advantage of this form of organization would be destroyed. . . ."[20]

As in any Prisoners' Dilemma game, the assumption of Riker and Brams that "motions are made and voted upon serially, so that future motions are not necessarily anticipated when current motions are considered"[21] does provide the potential for learning behavior over time. In repeated legislative sessions, members may begin to trust each other and pursue a strategy of cooperation rather than defection under a unanimity rule (as opposed to cooperation under majority rule) yielding the best possible outcome.[22] Yet the logical properties of the paradox remain intact and there is nothing to prevent a greedy member from attempting to maximize his own utility at the expense of the others, resulting again in a Prisoners' Dilemma and an occurrence of the paradox of vote trading. Thus, we have shown that the paradox is quite general, but that its effects are not neutral with respect to decision rules. Who bears the external costs depends upon who has the power to inflict them—the majority (in the case of majority rule) or the minority (under a unanimity rule).[23] On a more general level, however, we ask: Are the positions of Riker and Brams, on the one hand, and Buchanan, Tullock, and particularly Coleman on the other hand, necessarily in total conflict? It is to this question—and its implications for the relationship between vote trading and decision rules—that we now turn.

Paradox Gained and Paradox Lost

The initial discussion of the paradox by Riker and Brams suggests that they have uncovered an anomaly in the structure of voting procedures: the initial assumption is that vote trading should produce Pareto optimal results, but the paradox is established when the condition of vote trading is not met. Riker and Brams argue, however:

> . . . the paradox of vote trading is closely related to the paradox of voting. . . . [In the above example] no trade is preferred to all other trades, which is equivalent to our earlier demonstration that no coalition can defeat all others. Since there is no preferred set of trades on which to base a stable coalition, no stable coalition is possible. The paradox of vote trading cannot, therefore, be simply solved by waving it away with a coalition. Rather it is inherent in the nature of the legislative process and, given an appropriate distribution of tastes and external costs, cannot be avoided.[24]

Similarly, David H. Koehler has presented a proof that a cyclical majority will occur if and only if

[20] E. E. Schattschneider, *The Semisovereign People* (New York: Holt, Rinehart and Winston, 1960), p. 35. The paradox—at least as developed so far—follows rather directly from the fact that each voter can be said to be in either a winning coalition (for each motion or for the entire set of motions) or a losing coalition. Thus, the set of motions, *J*, constitutes a simple game: formally, "[*a*] game, *v*, in (0, 1) normalization is said to be simple if, for each $S \subset N$, we have either $v(S) = 0$ or $v(S) = 1$" (Guillermo Owen, *Game Theory* [Philadelphia: W. B. Saunders, 1968], p. 163), where N is the set of players in the game, *S* is a coalition, and $v(S)$ is the value of the coalition. Owen notes (*ibid.*) that "simple games . . . include voting 'games' in elections and legislatures"—precisely the situation we have. The result that at least one member of the total number of players must be excluded from some coalition follows from an extension of L. S. Shapley's comment that "[s]olutions that exclude one or more players . . . are found . . . in almost all simple games, and they probably exist in all constant-sum games." See Shapley, "On Solutions that Exclude One or More Players," in *Essays in Mathematical Economics in Honor of Oskar Morgenstern,* ed. Martin Shubik (Princeton: Princeton University Press, 1967), p. 57.

[21] Riker and Brams, "The Paradox of Vote Trading," p. 1236.

[22] Cf. Anatol Rapoport and Albert M. Chammah, *Prisoners' Dilemma* (Ann Arbor: University of Michigan Press, 1965), chap. 5; and Lester B. Lave, "An Empirical Approach to the Prisoner's Dilemma Game," *Quarterly Journal of Economics,* 76 (August, 1962), pp. 424–436. Nigel Howard has proposed an approach to the Prisoners' Dilemma in which cooperative strategies would be rational. In his "metagames," players choose their strategies dependent upon those of the other players. See his *Paradoxes of Rationality: Theory of Metagames and Political Behavior* (Cambridge: MIT Press, 1971), pp. 44–48 and 55–60. Note the strong criticism of this approach by John C. Harsany, in his review of Howard's book in the *American Political Science Review,* 67 (June 1973), 599–600. In experiments with repetitive play, individuals have tended to behave cooperatively. Rapoport and Chammah as well as Lave found that the proportion of cooperative responses approach seventy per cent. For an interesting argument that the Prisoners' Dilemma is not a genuine dilemma or even a paradox, see R. L. Cunningham, "Ethics and Game Theory: The Prisoners' Dilemma," *Papers on Non-Market Decision Making,* 2 (1967) pp. 11–26.

[23] The logical nexus between unanimity and majority voting rules vanishes once the deterministic nature of the argument is relaxed. The probability that an individual will agree to trade votes with one or more others is dramatically affected by the fact that, under the former decision rule, a single negative vote will suffice to defeat a motion. Under majority rule, however, the probability that a member will be able to affect the outcome of a collective decision is substantially reduced—and, indeed, is a function of the size of the voting body. Another factor which is also a function of the size of the voting body—and which seems to us to be more serious for a unanimity than a majority rule—is "transaction costs," associated with the necessity of reaching a collective decision. Our discussion of pairwise vs. n-wise trading of votes below bears on this problem. We are grateful to T. Nicolaus Tideman and James M. Buchanan for calling this general problem to our attention, although we emphasize that our approach is not always consistent with theirs (in private communications). Cf. also note 34 below.

[24] Riker and Brams, "The Paradox of Vote Trading," pp. 1245–1246.

the conditions for vote trading established by Riker and Brams and discussed above are met.[25] Note, however, the statement by Riker and Brams that "an appropriate distribution of tastes and external costs" must occur. Perhaps a reconciliation of the position espoused by Buchanan, Tullock, and Coleman and that of Riker and Brams and Koehler can be achieved. A critical first step in this process is to realize that the Riker-Brams discussion of the paradox of vote trading when preferences are ordinarily ranked is itself based upon an "appropriate" distribution of tastes and external costs. The salience measure they propose for their ordinal treatment of the paradox, $s_i(x)$, is itself a function of the associated cardinal utilities attached by an individual to the passage and failure of a given motion. Let $u_i(P_x)$ be the utility to Member i derived from the outcome associated with his preferred position on motion x and let $u_i(P_x)$ be a similar utility for i derived from the outcome for his not preferred position on x. Then the salience of the motion is defined as: $s_i(x) = |u_i(P_x) - u_i(\tilde{P}_x)|$, which is clearly a function of the cardinal properties of the associated utilities.[26]

Alterations in these cardinal utilities thus *may* entail alterations in the relative salience of the various motions, with the prospect that an "appropriate" distribution of tastes is not obtained. To take two examples,[27] in Table 1 above (a) consider the impact of changing the values of each motion on which the cardinal utilities are currently $(1, -2)$ to $(2, -1)$; or (b) consider a change from each entry of $(-2, 2)$ to $(-3, 3)$. Under the first change, a majority voting rule would yield each member a net utility of zero without vote trading and $+2$ if all players traded votes. The second proposed alteration of utilities in Table 1 would produce a unanimity rule outcome of -2 to each member without vote trading and of zero with logrolling. In both cases, the paradox is seemingly avoided. How can this occur?

The reason the paradox might be avoided under certain circumstances is that the distribution of preferences has a direct and identifiable impact on the vote trading situation. A second factor which must not be overlooked is that the ordinal properties of the paradox of voting can be maintained even if there are changes in the cardinal numbers associated with the specific outcomes in a chart of preferences such as Table 1. The change from utilities of $(-2, 2)$ to $(-3, 3)$ does not affect the ordinal rankings of the outcomes at all with respect to producing a paradox of voting. Coleman's critical point is that there is a difference between merely noting that one is in a situation in which the paradox of voting occurs and attempting to devise a strategy to extricate oneself from such a situation.[28] On the other hand, we must realize that at a fundamental level the paradox of vote trading is indeed inescapable. As long as individual trades are rational, then we have a cyclical pattern of vote trading which can never be stable, as the Riker-Brams analysis demonstrates.[29] This result is *not* a function of the associated cardinal utilities which members attach to outcomes. It thus should be cautioned that we are not proposing a logical way out of the paradox (although a very restrictive one will follow from a theorem we propose), but rather a practical way out. In particular, we argue that there are basically three types of situations: those in which sincere voting is Pareto optimal, those in which vote trading (sophisticated voting) is Pareto optimal, and those in which no decision rule and no voting strategy can improve upon a preordained societal result. In any of these situations, strategies which yield less than an optimum may indeed be reached. Our concern, however, is to distinguish the conditions under which the paradox of vote trading will occur and those under which vote trading will yield a Pareto optimal solution. In order to make this distinction, we need a set of cardinal utility measures.

We thus propose such a set: $u(x_{ij})$ for a set of voters $i \in I$ and a set of motions $j \in J$ where $u(x_{ij})$ is the utility of the passage of the jth motion to the ith voter: and $u(\bar{x}_{ij})$ where the various subscripts are as before and $u(\bar{x}_{ij})$ is the utility of the defeat of such motions. We offer the following theorem to provide necessary and sufficient conditions for the optimality of vote trading:

Theorem

(1) Sincere voting is Pareto optimal under majority rule if and only if:

$$\sum_i \sum_j u(x_{ij}) > \sum_i \sum_j u(\bar{x}_{ij}).$$

(2) The strategy adopted (sincere vs. sophisticated voting) is neutral under majority rule if and only if:[30]

[25] Koehler, "Vote-Trading and the Voting Paradox." See Appendix II.

[26] Riker and Brams, "The Paradox of Vote Trading," p. 1239.

[27] We are grateful to David H. Koehler (private communication) for suggesting the first example and to an anonymous referee for suggesting the second.

[28] Coleman, "The Possibility of a Social Welfare Function," especially p. 1107.

[29] Riker and Brams, "The Paradox of Vote Trading," p. 1244.

[30] Cf. Buchanan and Tullock, *The Calculus of Consent* (p. 145): "Potentially, the voter should enter into bargains until the marginal 'cost' of voting for something of which he disapproves but about which his feelings are weak exactly matches the expected marginal benefits of the vote or votes secured in return support for issues in which he is more interested."

$$\sum_i \sum_j u(x_{ij}) = \sum_i \sum_j u(\bar{x}_{ij}).$$

(3) Vote trading is Pareto optimal under majority rule if and only if:

$$\sum_i \sum_j u(x_{ij}) < \sum_i \sum_j u(\bar{x}_{ij}).$$

In the first appendix to this article, we present the proofs of the above optimality conditions. As formulated above, these proofs depend upon the assumption either that all motions pass or that all motions fail. We shall continue to employ this assumption in our discussion of the optimality conditions below because of its interesting consequences for "porkbarrel" projects on which legislators may trade votes. This restriction, however, can easily be lifted and our results generalized to situations in which some motions pass and others fail. What is critical here is that the relationships we derive are based upon the positions taken by the majority of legislators, on the one hand, and the minority of legislators, on the other hand.[31] Note that in part (1) of the theorem, Pareto optimality is itself defined by

$$\sum_i \sum_j u(x_{ij})$$

since this summation provides the boundaries for the production possibility curve in terms of the total utility for the society. Similarly, Pareto optimality for condition (2) can be represented by either

$$\sum_i \sum_j u(x_{ij}) \quad \text{or} \quad \sum_i \sum_j u(\bar{x}_{ij}),$$

while for condition (3), the optimality condition obviously is represented by

$$\sum_i \sum_j u(\bar{x}_{ij}).$$

We note that in the theorem, we do not restrict the trading situation to pairwise trades, as do Riker and Brams. The set of trading partners can be as large as the entire membership of the legislative body. This lack of restriction to pairwise trades will have interesting consequences for analyzing the paradox of vote trading under different situations of Pareto optimality.

We state a corollary of the theorem without proof: Under condition (1) vote trading is Pareto optimal under a unanimity rule; under condition (2) the strategy adopted under a unanimity rule is neutral; and under condition (3) sincere voting is Pareto optimal under a unanimity rule. The last part of the corollary is perhaps the least obvious. In our discussion above, we noted that the effects of vote trading under a unanimity rule are indeed just the opposite of those under majority rule. Therefore, if vote trading is optimal under majority rule for condition (3), sincere voting would be expected to be (and indeed is) optimal under a unanimity rule.

Conclusions

The appropriate distribution of tastes and externalities for the optimality of sincere voting, then, is simply that the voters collectively have more to gain by the passage of all the motions than they do by defeating them. This result is heuristic: majority rule without vote trading will produce a Pareto optimal result under such conditions. If the voters attach equal utility across the set of motions to their passage or their defeat, then no decision rule or strategic premise is going to change the net utility of the outcome. Finally, if in the aggregate, the voters attach more utility to the defeat of all motions than they do to passing all of them, then majority rule with vote trading (or a unanimity rule without logrolling) can yield a Pareto optimal result. Only, however, in the very restrictive case in which the size of the trading set is equal to the number of legislators will the paradox of vote trading be avoided in a formal sense. In this situation, we have a totally cooperative *n*-person game in which the enforcement power has been agreed upon by the players in advance. More likely situations involve either pairwise trades only—a noncooperative game—as developed by Riker and Brams and also by Gerald H. Kramer, or at best only a partially cooperative game (in which trading coalitions may be greater than pairwise in size but smaller than the entire chamber), as John A. Ferejohn has argued.[32] Thus, the optimality of vote trading under condition (3) does not extricate us from the paradox of vote

[31] This condition together with the condition for vote trading introduced in the proofs below, is based upon situations in which (under majority rule) all motions will pass. While seemingly restrictive, the same logic underlying the three proofs can easily be extended to "corollaries" assuming that all motions will fail or that some will pass and some will fail. A less cumbersome approach, however, can extend the theorem we have presented rather easily. Let $u(x_{ij})$ represent the utility to legislator i of the *majority* position on issue j and $u(\bar{x}_{ij})$ represent the utility to i of the *minority* position on j (irrespective of the nature of the actual outcomes). Then the theorem we propose is completely general and subsumes not only our special case but all of the possible "corollaries" representing alternative cases.

[32] See Riker and Brams, "The Paradox of Vote Trading," pp. 1238ff.; Kramer, "Sophisticated Voting over Multidimensional Choice Spaces," *Journal of Mathematical Sociology*, 2 (July, 1972), 125–180; and Ferejohn, "Sour Notes on the Theory of Vote Trading," California Institute of Technology, Social Science Working Paper Number 41 (June 1, 1974). We are particularly grateful to Steven J. Brams for his comments on this aspect of the present work.

trading unless we make the very strong assumption that the trading game is totally cooperative. On the other hand, all is not lost because *in actual legislative situations*, we would expect that "learning behavior" could easily provide the basis for legislators to extricate themselves from the paradox of vote trading. And, indeed, to the extent that legislators are rational actors, we should expect that they will indeed do so.

It is not terribly difficult for a rational legislator to realize when to stop trading and when the Pareto optimal position has been reached (particularly if he has been "stung" once by an occurrence of the paradox of vote trading). While Riker and Peter C. Ordeshook have argued that in most large legislatures members are likely to realize the potentially harmful effects of vote trading (and, hence, restrict any trading activity to a smaller protocoalition such as fellow party members),[33] evidence from the American Congress suggests that "learning behavior" is indeed evident on the optimality of vote trading on "porkbarrel" votes. As James T. Murphy has commented:

> [On the House Public Works Committee] . . . the two parties *distrust* each other. After all, public works projects are inherently political and, as such, are presumed to have substantial electoral significance. Since each party values majority status, each expects the other to porkbarrel . . . . when the other is in the majority. Because they foresee the possibility of partisan allocations and because they take the allocation of public works benefits so seriously, congressmen insist on a fair allocation of the goodies. Hence, they have agreed upon permanent ways to reduce electoral risks when a majority in each party is affected by an allocation. . . . . *Only when a majority in each party benefits from an allocation can party cooperation be expected.* Since committee Democrats and Republicans are confronted with exactly the same situation as House Democrats and Republicans, the electoral risk-reducing devices explain party cooperation in the committee as well as in the House.[34]

[33] Riker and Ordeshook, *An Introduction to Positive Political Theory* (Englewood Cliffs, N.J.: Prentice-Hall, 1973), pp. 113–114. Note especially their reference to James Murphy, *The Empty Porkbarrel* (Lexington, Mass.: D. C. Heath, forthcoming), which (according to Riker and Brams) argues that very little logrolling actualy takes place even on public works legislation. Also, cf. David R. Mayhew, *Party Loyalty among Congressmen* (Cambridge: Harvard University Press, 1966: ch. 6) on the optimality of intra-party as opposed to inter-play cooperation. And, cf. the comments made in President Gerald R. Ford's inaugural speech to a joint session of Congress on August 12, 1974. Addressing in particular House Speaker Carl Albert (D., Okla.), Ford stated: "I have sometimes voted to spend more taxpayers' money for worthy federal projects in Grand Rapids [Michigan] while vigorously opposing wasteful federal boondoggles in Oklahoma" [cited in *Congressional Quarterly Weekly Report,* 32 (August 17, 1974), 2209] Also cf. the comments of Martin and Susan Tolchin (*To the Victor* [New York: Vintage, 1972], p. 218): ". . . it was no surprise that when Representative Charles Joelson, a New Jersey Democrat, wrote a letter to each member of Congress asking 'Where can we economize in your district?' not one reply was returned to his office." Many members probably thought that the letter was just another example of the humor for which Joelson was known (and to which the first-named author of this article can attest). Further evidence for the claim that legislators may be aware of the paradox of vote trading can be inferred from Herbert B. Asher, "The Learning of Legislative Norms," *American Political Science Review,* 67 (June, 1973), 499–513. Asher notes (p. 508) that 68 per cent of a sample of 22 freshmen in the 91st House stated that they would be willing to trade votes with a colleague. While the effects of learning behavior do seem to be apparent in the House—more nonfreshmen (81 per cent of a sample of 21) expressed a willingness to trade than freshmen (p. 501)—the comment of a senior member is instructive (p. 503): "Yes, I'd trade votes, but this does not happen often. It is not a specific trade, but more a matter of good will. I never had a specific trade; this happens more often in the Senate." (On the utility of trading through "good will," or implicit logrolling, see Appendix II.)

[34] Murphy, "Political Parties and the Porkbarrel: Party Conflict and Cooperation in House Public Works Committee Decision-Making," *American Political Science Review,* 68 (March 1974), 179–180 (first emphasis in original; second emphasis added). We thus find ourselves in the interesting situation of citing a single author on two different sides of the same question (cf. note 33 above). We await publication of his book before pursuing the interpretation of Riker and Ordeshook any further. For similar comments, see Robert F. Fenno, Jr., *Congressmen in Committees* (Boston: Little, Brown, 1973), pp. 58, 156–159, and 165–166; and Mayhew, *Congress: The Electoral Connection* (New Haven: Yale University Press, 1974), pp. 86–91. For a somewhat different perspective on the effects of vote trading for porkbarrel projects from that presented in the text, see Ferejohn, "Sour Notes on the Theory of Vote Trading," p. 11. For a statement which is more in accord with our view of the logrolling problem and also treats the problem of transaction costs (cf. note 23 above), see Mancur Olson, Jr., "The Optimal Allocation of Jurisdictional Responsibility: The Principle of 'Fiscal Equivalence'," in United States Congress, Joint Economic Committee, 91st Congress, First Session, *The Analysis and Evaluation of Public Expenditures: The PPB System,* I (Washington: Government Printing Office, 1969), pp. 321–331. Olson argues (p. 326): "If all mutually advantageous bargains were struck, logrolling would insure that all collective goods that it was Pareto-optimal to provide would be provided. But . . . especially where large groups of people are at issue, it will very often be the case that logrolling will not happen, and that there will not be a Pareto-optimal supply of public goods." He adds (p. 326, n. 14): "In the United States Congress, logrolling probably leads to a greater expenditure when projects of a 'pork barrel' type are at issue. In most of these cases the projects are of a tangible, if not monumental type, and a Congressman is more likely to be identified in his district with such a project than with a general tax increase, which could not in any case usually be traced to any one package of local projects." For an argument similar to that of Olson on government expenditures, see J. Ronnie Davis and Charles W. Meyer, "Budget Size in a Democracy," *Southern Economic Journal,* 36 (July, 1969), 10–17.

Thus, on issues in which individual rationality might lead a member to vote against a porkbarrel project for another member (and, particularly, one from the opposition party), there still might be a collective rationality in voting for the project —and members seem to be able to "learn" how to resolve the formal problem of pairwise trading instability.

The result is traceable to the fact that straightforward majority rule nevertheless will produce majorities for all six motions under condition (3); this voting procedure is insensitive to variations in individual cardinal utilities. The conditions for the optimality of vote trading, however, are built around assumptions about cardinal utilities. When the decision reached by counting votes is the same as that reached by counting units of utility—as in condition (1)—there is no conflict between the majority principle and the salience of individual issues. On the other hand, when the two criteria come into direct conflict, the mechanism of vote trading (or of a unanimity rule without vote trading, which also is constructed so as to protect "intense minorities") has the potential to avoid a paradox of vote trading.

In particular, we are intrigued by the failure of a unanimity rule with vote trading to dominate the results obtained for such a rule without logrolling. In the first case, where a supposed domination does occur, we encounter the paradox of vote trading. However, the advocates of a unanimity rule with vote trading—Buchanan and Tullock—were particularly concerned with situations in which an intense minority might be tyrannized by a relatively apathetic majority. It is precisely in this situation—where

$$\sum_i \sum_j u(x_{ij}) < \sum_i \sum_j u(\bar{x}_{ij})$$

that vote trading is Pareto optimal under majority rule but inferior to sincere voting for a unanimity rule. The paradox with which we are confronted, then, is over the effects of alternative decision rules and voting strategies on the problem of externalities. We do not have a unidirectional linkage between the rules and the strategies, but a more complicated web than we had imagined. Indeed, the only situation in which a unanimity decision rule with vote trading is Pareto optimal involves "intense majorities" rather than "intense minorities."

The relationship between the operative decision rule and the optimal voting strategy (if such a strategy exists) is itself a function of the distribution of preferences—and thereby affects the distribution of externalities. As Riker and Brams comment, the critical distinction is between the "market for votes" and the "market for goods": vote trading is an imperfect barter system in which only whole commodities (votes) rather than infinitely divisible ones (e.g., money) must be exchanged.[35] This feature of the imperfect vote market increases the effects of both the decision rule and the voting strategy on the magnitude of external costs and benefits. Only when externalities are effectively nil

$$\sum_i \sum_j u(x_{ij}) = \sum_i \sum_j u(\bar{x}_{ij})$$

does the relationship between voting strategies and decision rules cease to affect the total utility over the set of outcomes.

In the case of the paradox of vote trading, the imperfections of the market produce a dynamic relationship between decision rules and voting strategies which effectively prevents the attainment of Pareto optimality. Under the final set of conditions

$$\sum_i \sum_j u(x_{ij}) < \sum_i \sum_j u(\bar{x}_{ij}).$$

either majority rule with vote trading or a unanimity rule without logrolling can yield a Pareto optimal outcome. Thus, the two mechanisms suggested by Buchanan and Tullock both may work to reduce external costs, but not simultaneously. When a unanimity rule yields an optimal distribution of utilities, vote trading can at best fail to improve upon this distribution and at worst product a markedly suboptimal distribution of external costs, thereby leaving everyone worse off than if no votes had been cast at all.

Thus our analysis has demonstrated that the positions of Riker and Brams, on the one hand, and that of Buchanan, Tullock, and Coleman, on the other hand, are not necessarily in total conflict (although for rather different reasons from those Buchanan and Tullock posited). Except in the extreme case in which *all* legislators agree to trade (for majority rule), the paradox of vote trading cannot be avoided as a logical problem. This does not mean, however, that sincere voting is Pareto optimal under all circumstances. What we emphasize here is that Riker and Brams are concerned with a formal problem of voting games, whereas Buchanan, Tullock, and Coleman have stressed optimality conditions. Clearly, as we have demonstrated, the two situations are not logically equivalent. Which solution is more satisfactory depends upon the dynamics of an actual social situation and the answer to the fundamental question: how high are the costs of reaching a

[35] Riker and Brams, "The Paradox of Vote Trading," pp. 1235 and 1238. Cf. Coleman, "The Possibility of a Social Welfare Function: Reply," pp. 1315–1317.

collective decision (transactions costs) among all players?

Appendix I

In this appendix, we present a proof of the theorem (but not the corollary) stated in the text. We restate the definitions of $u(x_{ij})$ as the utility that the ith legislator receives from the passage of the jth motion and $u(\bar{x}_{ij})$ as the utility he receives from the defeat of such motions. Denote the total set of voters (on any motion) as I and partition the set I into the following disjoint subsets: A, the set of traders (on any motions) and B, the set of nontraders on all motions. We also restate the theorem:

Theorem

(1) Sincere voting is Pareto optimal under majority rule if and only if:

$$\sum_i \sum_j u(x_{ij}) > \sum_i \sum_j u(\bar{x}_{ij}),$$

(2) The strategy adopted (sincere vs. sophisticated voting) is neutral under majority rule if and only if:

$$\sum_i \sum_j u(x_{ij}) = \sum_i \sum_j u(\bar{x}_{ij}).$$

(3) Vote trading is Pareto optimal under majority rule if and only if:

$$\sum_i \sum_j u(x_{ij}) < \sum_i \sum_j u(\bar{x}_{ij}).$$

We shall also assume, without loss of generality, that

$$\sum_i \sum_j u(x_{ij}) = 0.$$

Given the definitions of Pareto optimality in the text, we immediately turn to the proof, which like the theorem itself, is in three parts.

First consider condition (1):

Let

$$\sum_i \sum_j u(x_{ij}) = 0 \text{ and } \sum_i \sum_j u(\bar{x}_{ij}) < 0$$

For trades to occur, we have:

$$\sum_{i\in A} \sum_j u(\bar{x}_{ij}) > 0.$$

For nontraders, by stipulation:

$$\sum_{i\in B} \sum_j u(x_{ij}) \leq 0.$$

Since the total utility to all players is assumed to be zero,

$$\sum_{i\in A} \sum_j u(\bar{x}_{ij}) + \sum_{i\in B} \sum_j u(x_{ij}) \leq 0.$$

To prove sufficiency, let

$$\sum_i \sum_j u(\bar{x}_{ij}) < \sum_i \sum_j u(x_{ij}) = 0.$$

If all members trade votes (i.e., B is empty):

$$\sum_{i\in A} \sum_j u(\bar{x}_{ij}) > \sum_i \sum_j u(x_{ij}).$$

But this contradicts the initial premise that

$$\sum_i \sum_j u(\bar{x}_{ij}) < \sum_i \sum_j u(x_{ij})$$

since, if B is empty

$$\sum_{i\in A} \sum_j u(\bar{x}_{ij}) > 0$$

$$\forall i \in A \rightarrow \sum_i \sum_j u(\bar{x}_{ij}) > 0.$$

But

$$\sum_i \sum_j u(\bar{x}_{ij}) < 0$$

by stipulation. Thus, B is non-empty and sufficiency is established.

The proof of necessity is not as direct because

$$\sum_i \sum_j u(x_{ij}) > \sum_i \sum_j u(\bar{x}_{ij})$$

does not follow directly from the fact that B is non-empty. As condition (2) of the theorem demonstrates, the non-emptiness of B is also consistent with

$$\sum_i \sum_j u(x_{ij}) = \sum_i \sum_j u(\bar{x}_{ij}).$$

If B is non-empty, then either (1) vote trading yields a net loss of utility from sincere voting; or (2) vote trading among all $i \in I$ produces no net change of utility across the motions $j \in J$; or (3) only a subset of I, $i \in A$, can trade profitably and

$$\forall i \in B, \qquad u(x_{ij}) < 0.$$

The condition for vote trading,

$$\sum_{i\in A} \sum_j u(\bar{x}_{ij}) > 0,$$

effectively rules out non-profitable trades among *all* $i \in I$, but it does not eliminate profitable trading among some subset of $A \subset I$. If such a subset can trade profitably, B may be non-empty and

$$\sum_i \sum_j u(x_{ij}) \geq \sum_i \sum_j u(\bar{x}_{ij}).$$

The suboptimality of vote trading, however implies that:

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) < \sum_{i \in B} \sum_j u(x_{ij})$$

which in turn implies that A is empty and B is non-empty (specifically, the condition for vote trading is violated). Since

$$\sum_{i \in B} \sum_j u(x_{ij}) > \sum_{i \in A} \sum_j u(\tilde{x}_{ij}),$$

$$\sum_{i \in B} \sum_j u(x_{ij}) > \sum_{i \in B} \sum_j u(\tilde{x}_{ij}).$$

Otherwise, vote trading would be profitable $\forall i \in B$. Since

$$\forall i \in I, \quad i \in B, \quad \sum_{i \in B} \sum_j u(x_{ij}) > \sum_{i \in B} \sum_j u(\tilde{x}_{ij}) \rightarrow \sum_i \sum_j u(x_{ij}) > \sum_i \sum_j u(\tilde{x}_{ij}),$$

establishing necessity.

The proof for the second condition is as follows:

Let

$$\sum_i \sum_j u(x_{ij}) = \sum_i \sum_j u(\tilde{x}_{ij}) = 0.$$

For trades to occur, we have:

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) > 0.$$

For the non-traders, by stipulation:

$$\sum_{i \in B} \sum_j u(x_{ij}) \leq 0$$

Thus

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) + \sum_{i \in B} \sum_j u(x_{ij}) \leq 0.$$

To prove sufficiency, let

$$\sum_i \sum_j u(x_{ij}) = \sum_i \sum_j u(\tilde{x}_{ij}) = 0.$$

Assume that B is empty. Then, by the condition for vote trading,

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) > \sum_i \sum_j u(x_{ij}) = 0 \rightarrow \sum_i \sum_j u(\tilde{x}_{ij}) > \sum_i \sum_j u(x_{ij}).$$

since $\forall i \in I$, $i \in A$. But this yields a contradiction since

$$\sum_i \sum_j u(x_{ij}) = \sum_i \sum_j u(\tilde{x}_{ij})$$

by assumption. Now assume that A is empty. Then, by the condition for vote trading,

$$\sum_{i \in B} \sum_j u(x_{ij}) \leq 0.$$

If the strict inequality holds,

$$\sum_{i \in B} \sum_j u(x_{ij}) < 0 \rightarrow \sum_i \sum_j u(x_{ij}) < 0$$

since $\forall i \in I$, $i \in B$. But, again, this yields a contradiction, implying that A is non-empty. Thus sufficiency is established.

To establish necessity, we demonstrate that: (1) if B is empty, vote trading is not profitable and (2) if A is empty, sincere voting would not improve upon the results of vote trading. Assume that B is empty. Then

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) \leq 0.$$

But this implies that

$$\sum_i \sum_j u(\tilde{x}_{ij}) \leq 0$$

since $\forall i \in I$, $i \in A$. However, the condition for vote trading is

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) > 0$$

and we have

$$\sum_i \sum_j u(\tilde{x}_{ij}) > 0.$$

But, by assumption

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) + \sum_{i \in B} \sum_j u(x_{ij}) \leq 0.$$

Thus, if B is empty, the condition for vote trading must be violated. Hence, B is non-empty. Now assume that A is empty. Then,

$$\sum_{i \in B} \sum_j u(x_{ij}) \geq \sum_i \sum_j u(\tilde{x}_{ij}) \rightarrow \sum_i \sum_j u(x_{ij}) \geq \sum_i \sum_j u(\tilde{x}_{ij})$$

since $\forall i \in I$, $i \in B$. If the strict inequality holds, we have:

$$\sum_i \sum_j u(x_{ij}) > \sum_i \sum_j u(\tilde{x}_{ij})$$

which implies that sincere voting would improve upon the results of vote trading. However,

$$\sum_i \sum_j (\tilde{x}_{ij}) = 0$$

by assumption and thus

$$\sum_i \sum_j u(x_{ij}) > 0:$$

This contracts our assumption that

$$\sum_{i \in B} \sum_j u(x_{ij}) \leq 0,$$

since $\forall i \in I, i \in B$. Thus.

$$\sum_i \sum_j u(x_{ij}) = \sum_i \sum_j u(\tilde{x}_{ij}),$$

establishing necessity.

Finally we have:

Let

$$\sum_i \sum_j u(\tilde{x}_{ij}) = 0 \text{ and } \sum_i \sum_j u(x_{ij}) < 0.$$

For trades to occur, we have:

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) > \sum_i \sum_j u(x_{ij}).$$

For nontraders. by stipulation:

$$\sum_{i \in B} \sum_j u(x_{ij}) < 0.$$

To establish that vote trading is profitable under these conditions—the proof of sufficiency—we must show: (1) A is non-empty and (2) B may be empty. We need not demonstrate that B is always empty since that would imply that the *only* profitable trading situation occurs when all players trade, which is not maintained here.

Thus, the conditions for trading and nontrading give:

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) + \sum_{i \in B} \sum_i u(x_{ij}) \leq 0.$$

Let

$$\sum_i \sum_j u(\tilde{x}_{ij}) = 0 \quad \text{and} \quad \sum_i \sum_j u(x_{ij}) < 0.$$

Assume first that A is empty. Then

$$\sum_{i \in B} \sum_j u(x_{ij}) = \sum_i \sum_j u(x_{ij}) < 0$$

since $\forall i \in I, i \in B$. If A is empty, however, the condition for vote trading

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) > \sum_i \sum_j u(x_{ij})$$

must be violated. The violation of the condition for vote trading, however, contradicts our initial premise that

$$\sum_i \sum_j u(\tilde{x}_{ij}) > \sum_i \sum_j u(x_{ij}).$$

Thus, A is non-empty. Now suppose that B is empty. Since A is non-empty,

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) \leq 0$$

and

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) > \sum_i \sum_j u(x_{ij}) \rightarrow \sum_i \sum_j u(\tilde{x}_{ij}) > \sum_i \sum_j u(x_{ij})$$

if B is empty. Since this result does not contradict our initial premise, B may be empty and sufficiency is established.

To prove necessity we must show that

$$\sum_i \sum_j u(\tilde{x}_{ij}) > \sum_i \sum_j u(x_{ij})$$

if: (1) A is non-empty and (2) B may be empty. If B is empty, then

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) = \sum_i \sum_j u(\tilde{x}_{ij}) > \sum_i \sum_j u(x_{ij})$$

by the condition for vote trading. Assume that A is empty. Then, the condition for vote trading must be violated and:

$$\sum_{i \in A} \sum_j u(\tilde{x}_{ij}) \leq \sum_i \sum_j u(x_{ij}) \rightarrow \sum_{i \in B} \sum_j u(x_{ij}) = \sum_i \sum_j u(x_{ij}) \geq \sum_i \sum_j u(\tilde{x}_{ij}).$$

Thus,

$$\sum_i \sum_j u(\tilde{x}_{ij}) > \sum_i \sum_j u(x_{ij})$$

only if A is non-empty. Necessity is established.

Appendix II

The distribution of tastes which Riker and Brams discuss (see note 24 above) is one which yields a cyclical majority. It can readily be confirmed that a cycle is found in Table 1 above. Particularly in the case of cyclical majorities, the strategy of implicit vote trading (always trading with the same partner) is not likely to yield optimal results. Implicit vote trading on a large scale implies a cohesive voting system (and the potential for responsible parties). Such a system is realizable only if the distribution is single-peaked. Riker and Brams ("The Paradox of Vote Trading," pp. 1244–1246) discuss this problem. A specific set of illustrations will help. Let $u_i(1, 2)$ denote the utility to voter i obtained from an implicit vote trading agreement between members 1 and 2, etc. Then, for simple majority rule, we have:

$u_1(1, 2) = 3$	$u_1(1, 3) = 3$	$u_1(2, 3) = -10$
$u_2(1, 2) = 3$	$u_2(1, 3) = -10$	$u_2(2, 3) = 3$
$u_3(1, 2) = -10$	$u_3(1, 3) = 3$	$u_3(2, 3) = 3$

For a unanimity voting rule, we have:

$$u_1(1, 2) = -1 \quad u_1(1, 3) = -1 \quad u_1(2, 3) = 8$$
$$u_2(1, 2) = -1 \quad u_2(1, 3) = 8 \quad u_2(2, 3) = -1$$
$$u_3(1, 2) = 8 \quad u_3(1, 3) = -1 \quad u_3(2, 3) = -1$$

In the case of majority rule, implicit trading is profitable for both trading partners, but no single coalition dominates any of the others. Since the expected value of the three coalitions is equal to -4, the three coalitions are not even preferable to the results obtained from the paradox of vote trading. None of these coalitions, however, is dominated by the results from simple majority rule without vote trading, so that all three possible coalitions remain in the core of the voting game. For a unanimity rule, note that the nontrader still benefits from the trading activity of the others. The $+8$ entries for the nontrader in each coalition are simply the values of his security level (see Tables 5 and 6 above). In each case, the traders are worse off than if no logrolling had occurred—but actually are better off than if they had attempted to employ "explicit" sophisticated voting strategies. But, the outcomes for sophisticated voting are themselves not in the core for the voting game under a unanimity rule, whereas a unanimity rule with vote trading dominates each of the three solutions for "implicit" vote trading. Thus, the coalitions from implicit vote trading are not in the core for a unanimity voting rule—and neither is the "actual" solution of the voting game: a refusal to trade by each player. This "solution" is in turn dominated by each incidence of implicit vote trading. The paradoxical result is that the only solution considered which is in the core is not attainable. Hence, we have the paradox of vote trading and the contradiction is even more clearly defined for a unanimity rule than it is for a majority decision rule (in which implicit vote trading solutions are indeed in the core). This is the case because, as we argue above (see note 19), the vote trading situation under a unanimity rule is a better illustration of the traditional Prisoners' Dilemma than is the situation under majority rule.